WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541, January, 2012

Show all your work. You may quote named results. Each part is worth 10 points.

1. Let X be a sample of size 1 from the distribution with probability mass function (pmf)

$$f(x|\theta) = \frac{1}{\theta},\ x = 1, 2, \ldots, \theta,\ \theta \in \{1, 2, 3, \ldots\}.$$

   (a) Show that the family of distributions is complete.

   (b) Find a UMVUE of θ.

2. Let X be a random variable whose pmf under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

   (a) Find the most powerful test for H_0 versus H_1 with size $\alpha = 0.04$.

   (b) Compute the probability of Type II error.

3. The double exponential distribution (also known as the Laplace distribution) has density function

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|},\quad -\infty < x < \infty, \text{where } \lambda > 0.$$

   Let $X_1, \ldots, X_n$ be a random sample from this distribution. Suppose it is desired to test $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

   (a) Derive a level α score test of H_0 versus H_1.

   (b) Derive a level α Wald test of H_0 versus H_1, based on the MLE as test statistic and using the observed information in computing its variance estimate. Comment on the differences between the score and Wald tests.

4. The guaranteed exponential distribution has density function $f(t) = \lambda e^{-\lambda(t-G)},\ t > G$, where $\lambda,\ G > 0$. Let $T_{(j)},\ j = 1, \ldots, n$ be the order statistics from a random sample of size n.

   (a) Show that $U = \sum_{i=2}^{n} T_{(i)}$ and $T_{(1)}$ are jointly sufficient for λ and G.

   (b) Find the maximum likelihood estimates of λ and G.

   (c) Show that $n(T_{(1)} - G)$, and $(n-j+1)(T_{(j)} - T_{(j-1)}),\ j = 2, \ldots, n$ are iid exponential random variables with mean $1/\lambda$.

   (d) Using the above results, show how to construct exact confidence intervals for λ and G using only the data and tables of the chi-square and F distributions.